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A Limit Set Trichotomy for Positive Nonautonomous Discrete Dynamical Systems

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For a sequence $(T_n)_{n \in \mathbb{N}}$ of self-mappings of a normal cone in a real Banach space, conditions are specified which imply a limit set trichotomy for the inhomogeneous iterates $T_n \circ T_{n-1} \circ \cdots \circ T_1 \circ T_0$. © 1999 Academic Press

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1. INTRODUCTION

This paper is concerned with the stability behavior of nonautonomous positive discrete dynamical systems

$$x_{n+1} = T_n x_n \qquad \text{for } n = 0, 1, 2, \ldots \text{ and } x_0 \in K, \tag{$*$}$$

where K is a normal cone of some real Banach space $(E, \|\cdot\|)$ and the operators T_n are continuous self-mappings of K. When the system $(*)$ is autonomous, that is when $T_n = T$ for all $n \geq 0$, Krause and Nussbaum [10] explored conditions on the self-mapping T of K which ensures that the following stability trichotomy holds. Either

(i) for every $x_0 \in K \setminus \{0\}$ the orbit $\{x_n | n \geq 0\}$ is an unbounded set; or

(ii) for every $x_0 \in K$, $\lim_{n \to \infty} \|x_n\| = 0$; or

(iii) there exists some x^* in the interior $\mathring{K}$ of K such that $Tx^* = x^*$ and $\lim_{n \to \infty} \|x_n - x^*\| = 0$ for all $x_0 \in K \setminus \{0\}$.

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This theorem is a generalization of results of Krause and Ranft [11] and Smith [18] for the finite dimensional case. Also Takác [19] generalized the result in [18]. Krause [9] gives an overview on these results and their relation plus further references.

In this paper we shall extend the above stated trichotomy to nonautonomous discrete dynamical systems. For nonautonomous systems, the stability result in case (iii) of the limit set trichotomy is weaker than in the autonomous case in the sense that only

$$\lim_{n\to\infty} \|x_n - x'_n\| = 0$$

holds for every two solutions $(x_n)_{n\geq 0}$ and $(x'_n)_{n\geq 0}$ of the nonautonomous system $(*)$. This *stability of motion* may occur even if each single solution shows chaotic behavior or eventually becomes periodic. That kind of stability is important for numerous applications in biology, demography, and economics, where dynamic processes are modeled frequently as time-discrete nonautonomous dynamical systems (cf. [2, 3, 12, 15, 17]).

It turns out that the system $(*)$ shows such a limit set trichotomy if the sequence of operators $(T_n)_{n\geq 0}$ is contractive with respect to the *part-metric* or Thompson's metric (cf. [1, 6, 10, 14, 20]) which will be defined below. Furthermore, we establish additional statements for nonautonomous systems, for which the operators T_n converge uniformly on K to an operator $T\colon K \to K$. As a special case, we will obtain the above mentioned trichotomy for autonomous systems in [10]. Finally, we will give two conditions for the operators T_n which ensure contractiveness with respect to the part-metric. Under these conditions the limit set trichotomy can be applied to nonautonomous difference equations of higher order, e.g. see Krause [8] for the autonomous case, and Kocic and Ladas [4, 5] for related results.

2. STABILITY TRICHOTOMY FOR DISCRETE DYNAMICAL SYSTEMS

In the most general setting throughout this paper we consider self-mappings of normal cones K in a real Banach space $(E, \|\cdot\|)$ with nonempty interior. Recall that a *cone* K is a closed convex set $K \subset E$ such that $\lambda \cdot K \subset K$ for all $\lambda \geq 0$ and $K \cap -K = \{0\}$. Denote by $\mathring{K}$ the interior and $\overline{K}$ the closure of K. We write further $K^* := K - \{0\}$ for the pointed cone. The cone K induces a *partial ordering* on K by defining $x \leq y$ and $x < y$, if $y - x \in K$ and $y - x \in \mathring{K}$, respectively. A cone K is called *normal* if there exists a constant M such that $\|x\| \leq M\|y\|$ for all $x, y \in K$ with $x \leq y$. Without loss of generality let $M = 1$; i.e., the norm $\|\cdot\|$ is assumed to be monotone on K; otherwise an equivalent norm on E can be found for which $M = 1$ (e.g., see [16]).

The discrete dynamical system $(*)$ is completely described by the inhomogeneous iterates $T_n \circ \cdots \circ T_0$ of the sequence $(T_n)_{n \geq 0}$, because $x_n = T_{n-1} \circ \cdots \circ T_0 x_0$ for all $n \geq 0$. Hence, for every $x \in K$ the set

$$\gamma^+(x, T_n) = \gamma^+(x) := \bigcup_{n \geq 0} (T_n \circ \cdots \circ T_0 x)$$

is the positive semiorbit of x with respect to $(T_n)_{n \geq 0}$ and

$$\omega(x, T_n) = \omega(x) := \bigcap_{n \geq 0} \overline{\bigcup_{k \geq n} (T_k \circ \cdots \circ T_0 x)}$$

is the omega limit set of x.

Let (X, d) be a complete metric space. A mapping $T\colon X \to X$ will be called *nonexpansive* (with respect to d), if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in X$, and T will be called *contractive* (with respect to d), if $d(Tx, Ty) < d(x, y)$ for all $x, y \in X, x \neq y$. A *sequence* $(T_n)_{n \geq 0}$ of self-mappings of X will be called *uniformly contractive* (in contrast to asymptotic contractive sequences considered in [3]), if there exists a continuous function $c\colon X \times X \to \mathbb{R}_+$ such that the following two conditions are satisfied:

(a) $c(x, y) < d(x, y)$ for all $x, y \in X$, $x \neq y$ and

(b) $d(T_n x, T_n y) \leq c(x, y)$ for all $n \geq 0$, all $x, y \in X$.

Though the stability results are stated in the norm topology on K, our contractivity conditions for the sequence $(T_n)_{n \geq 0}$ on K will be formulated in a different topology: if $\lambda(x, y) := \sup\{\lambda \geq 0 | \lambda x \leq y\}$ for $x, y \in K^*$, then

$$p(x, y) := -\log \min\{\lambda(x, y), \lambda(y, x)\}$$

defines a quasimetric on K^* (because $p(x, y) \in [0, +\infty]$) and p is called *part-metric*. By $x \sim y$ iff $p(x, y) < \infty$ an equivalence relation is defined on K^*, the classes of which are called *parts of* K. It is easy to verify that p is a metric on $\mathring{K}$; moreover, Thompson [20] proved that $(\mathring{K}, p)$ is a complete metric space if K is a normal cone. (For the part-metric and related results see [1, 7, 10, 14, 20] and the references given there.)

It is worthwhile to investigate sequences $(T_n)_{n \geq 0}$ of self-mappings on K being uniformly contractive for the part-metric, because part-metric and norm-distance are related by the following inequalities [10, Lemma 2.3]:

LEMMA 1. *Let K be a cone in a real Banach space $(E, \|\cdot\|)$.*

(i) *If $x, y \in \mathring{K}$ and $\eta > 0$ is a real number such that $\overline{B(x, \eta)} \subset \mathring{K}$ and $\overline{B(y, \eta)} \subset \mathring{K}$ then*

$$p(x, y) \leq \log\left(1 + \frac{\|x - y\|}{\eta}\right).$$

(ii) *If K is normal with monotone norm, then for all $x, y \in K^*$,*

$$\|x - y\| \le (2e^{p(x,y)} - e^{-p(x,y)} - 1) \cdot \min\{\|x\|, \|y\|\}.$$

With the help of these metric preliminaries we are able to prove a limit set trichotomy for the nonautonomous system $(*)$.

THEOREM 2. *Let K be a normal cone in a real Banach space $(E, \|\cdot\|)$ with nonempty interior. Let $(T_n)_{n \ge 0}$ be a sequence of continuous self-mappings of K with $T_n(\mathring{K}) \subset \mathring{K}$ for all $n \ge 0$. Assume further that for an integer $r \ge 1$ the sequence $(S_n)_{n \ge 0}$ of lumped operators $S_n = T_{n+r-1} \circ \cdots \circ T_n$ is uniformly contractive on $\mathring{K}$ for the part-metric and that $S_n(K^*) \subset \mathring{K}$ for all $n \ge 0$. For all bounded orbits $\gamma^+(x, T_n)$, $x \in K$, assume that the closure is compact in K in the norm topology. Then the following trichotomy holds: either*

(i) *for all $x \in K^*$ the orbits $\gamma^+(x, T_n)$ are unbounded in norm; or*

(ii) *for all $x \in K$, $\lim_{n\to\infty} \|T_n \circ \cdots \circ T_0 x\| = 0$; or*

(iii) *for all $x \in K^*$ the orbits $\gamma^+(x, T_n)$ are bounded in norm, and the omega limit sets $\omega(x, T_n)$ are not empty and have a non-trivial clusterpoint.*

If $\omega(x, T_n) \subset \mathring{K} \cup \{0\}$ for all $x \in K^*$, then in case (iii) we additionally have

$$\lim_{n\to\infty} \|T_n \circ \cdots \circ T_0 x - T_n \circ \cdots \circ T_0 y\| = 0 \qquad \text{for all } x, y \in K^*. \qquad (\dagger)$$

Proof. Assume that (i) does not hold. We show first that in this case the orbit $\gamma^+(x, T_n)$ is bounded in norm for all $x \in K$ and either (ii) or (iii) holds. Secondly, we show that $(\dagger)$ holds in case (iii) under the additional assumption. For every $x \in K$ let $(x_n)_{n \ge 0}$ be the solution of $(*)$ with $x_0 = x$.

(1) If (i) holds not, then some $x \in K^*$ exists such that the orbit $\gamma^+(x, T_n)$ is bounded in norm; i.e., $\|x_n\| \le M$ for some constant $0 \le M < \infty$ and all $n \ge 0$. Let now $y \in K^*$ be arbitrary and let $(y_n)_{n \ge 0}$ be the solution with $y_0 = y$. By $S_n(K^*) \subset \mathring{K}$ and $T_n(\mathring{K}) \subset \mathring{K}$, it follows that $x_n, y_n \in \mathring{K}$ for all $n \ge r$. Thus, by the definition of the part-metric,

$$\beta := \max\{p(x_n, y_n) : r \le n \le 2r - 1\} < \infty.$$

By uniform contractivity, the lumped operators S_n are nonexpansive on $\mathring{K}$ for all $n \ge 0$ and so we obtain $p(x_n, y_n) \le \beta$ for all $n \ge r$. Now Lemma 1 (ii) implies that

$$\|x_n - y_n\| \le (2e^{\beta} - e^{-\beta} - 1) \cdot \|x_n\| \qquad \text{for all } n \ge r.$$

Since $\|x_n\| \le M$ for all $n \ge 0$, this implies that also $\gamma^+(y, T_n)$ must be bounded in norm. Hence, $\gamma^+(x, T_n)$ is bounded in norm for all $x \in K$. Since bounded orbits are relatively compact in K, it follows that $\gamma^+(x, T_n)$ contains converging sequences showing that $\omega(x, T_n) \neq \emptyset$ for all $x \in K$.

(2) Let $x, y \in K^*$ be arbitrary. Instead of the solutions $(x_n)_{n \ge 0}$ and $(y_n)_{n \ge 0}$ of $(*)$ with $x_0 = x$ and $y_0 = y$, we consider for $0 \le s \le r-1$ the subsystems

$$u_{n+1,s} = f_{n,s} u_{n,s} \quad \text{with} \quad u_{0,s} := x_s$$

and

$$v_{n+1,s} = f_{n,s} v_{n,s} \quad \text{with} \quad v_{0,s} := y_s,$$

where $f_{n,s} := S_{n \cdot r + s}$. Obviously, $u_{n,s} = x_{n \cdot r+s}$ and $v_{n,s} = y_{n \cdot r + s}$ for all $n \ge 0$, all $0 \le s \le r-1$. By assumption, the sequences $(f_{n,s})_{n \ge 0}$ are uniformly contractive for all s.

Let $0 \le s \le r-1$ be arbitrary, but fixed; then, for the sake of simplicity, we rewrite $u_{n,s}, v_{n,s}, f_{n,s}$ by u_n, v_n, f_n, respectively. By (1) the closures of $\gamma^+(u_0, f_n)$ and $\gamma^+(v_0, f_n)$ are compact in K as closed subsets of compact sets. Then for all infinite subsequences $(u_{m(n)})_{n \ge 0}$ and $(v_{m(n)})_{n \ge 0}$ of $(u_n)_{n \ge 0}$ and $(v_n)_{n \ge 0}$, respectively, we can find further subsequences $(u_{k(n)})_{n \ge 0}$ and $(v_{k(n)})_{n \ge 0}$ such that $\lim_{n \to \infty}(u_{k(n)}, v_{k(n)}) = (u^*, v^*) \in K \times K$ in the norm topology. By $S_n(K^*) \subset \mathring{K}$ and $T_n(\mathring{K}) \subset \mathring{K}$, it follows that $x_n, y_n \in \mathring{K}$ for all $n \ge r$, and thus $u_n, v_n \in \mathring{K}$ for all $n \ge 1$. Therefore, we have $p(u_1, v_1) < \infty$ and by the uniform contractivity of $(f_n)_{n \ge 0}$ it follows that

$$p(u_{n+1}, v_{n+1}) \le c(u_n, v_n) \le p(u_n, v_n) \le \cdots \le p(u_1, v_1) \qquad (\ddagger)$$

for all $n \ge 1$. This implies by the definition of the part-metric that u^* and v^* are in the same part of K. Hence, $u^* \neq 0$ implies that $v^* \neq 0$; thus, $\omega(\xi, T_n) = \{0\}$ for some $\xi \in K^*$ implies that $\omega(x, T_n) = \{0\}$ for all $x \in K^*$. So we proved that either (ii) or (iii) is true, where for case (ii) it remains to show that also $\omega(0, T_n) = \{0\}$. To this end, assume that $T_{m_0}(0) \neq 0$ for some $n_0 \ge 0$; otherwise, there is nothing to show. By a redefinition of the first n_0 operators T_n if needed, we can assume that $T_0(0) \neq 0$ without loss of generality. Then it follows that $T_{r+1} \circ \cdots \circ T_0(0) \in \mathring{K}$, which implies $\omega(0, T_n) = \{0\}$ by the same argumentation used above for $x_0, y_0 \in K^*$.

(3) Assume that neither (i) nor (ii) holds; then (iii) holds by (1) and (2). It remains to show (†) under the additional condition $\omega(x, T_n) \subset \mathring{K} \cup \{0\}$ for all $x \in K^*$. Let $x_0, y_0 \in K^*$ be arbitrary, and for arbitrary, but fixed $0 \le s \le r-1$ let $(u_n)_{n \ge 0}$ and $(v_n)_{n \ge 0}$ denote the sequences defined

in (2). We show that $\lim_{n\to\infty}\|u_n - v_n\| = 0$. As s is chosen arbitrarily, this implies $\lim_{n\to\infty}\|x_n - y_n\| = 0$.

By (2) we know that $\lim_{n\to\infty}(u_{k(n)}, v_{k(n)}) = (u^*, v^*) \in K \times K$ in the norm topology for some increasing sequence $(k(n))_{n\geq 0}$ of natural numbers, where the cluster points u^* and v^* are in the same part of K. We can assume that $u^*, v^* \in K^*$ since (ii) holds not. Observe that $u^* = v^*$ for all such sequences $(k(n))_{n\geq 0}$ concludes the proof. Obviously, (‡) implies that

$$\lim_{n\to\infty} p(u_n, v_n) = \lim_{n\to\infty} c(u_n, v_n) = p^* \geq 0.$$

We must show that $p^* = 0$. Since $\omega(u_0, f_n) \subset \omega(x_0, T_n) \subset \mathring{K} \cup \{0\}$ and $\omega(v_0, f_n) \subset \omega(y_0, T_n) \subset \mathring{K} \cup \{0\}$ by assumption, it follows that $u^*, v^* \in \mathring{K}$. We claim that the part-metric p is coordinate-wise continuous in (u^*, v^*) in the norm topology. Then it is true that

$$p^* = \lim_{n\to\infty} p(u_{k(n)}, v_{k(n)}) = p(u^*, v^*),$$

and therefore, $c(u^*, v^*) = p^*$, because c is continuous with respect to the part-metric which is continuous in (u^*, v^*) in the norm topology. Then $u^* \neq v^*$ yields the contradiction

$$p^* = c(u^*, v^*) < p(u^*, v^*) = p^*$$

according to the definition of the function c. Hence, $u^* = v^*$.

It remains to show that the part-metric p is coordinate-wise norm-continuous in $(u^*, v^*) \in \mathring{K} \times \mathring{K}$. Obviously, we can find some $\eta > 0$ such that $\overline{B(u^*, \eta)} \subset \mathring{K}$ as well as $\overline{B(v^*, \eta)} \subset \mathring{K}$. For arbitrary, sufficiently small $\varepsilon > 0$ choose $\delta = \eta(e^{\varepsilon/2} - 1) > 0$, then $\|u^* - u\| + \|v^* - v\| < \delta$ implies by Lemma 1 (i) that

$$\begin{aligned} |p(u^*, v^*) - p(u, v)| &\leq p(u^*, u) + p(v, v^*) \\ &\leq \log\left(1 + \frac{\|u^* - u\|}{\eta}\right) + \log\left(1 + \frac{\|v^* - v\|}{\eta}\right) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

∎

Remarks. 1. Theorem 2 extends Theorem 3.1 in [10] to nonautonomous systems in a natural way: for autonomous systems $T_n = T$ for all $n \geq 0$ in $(*)$ such that $T^r(K^*) \subset \mathring{K}$ as given in [10, Theorem 3.1] it is evident that $\omega(x, T) = T(\omega(x, T))$ for all $x \in K$; hence, $\omega(x, T) = T^r(\omega(x, T)) \subset \mathring{K} \cup$

$\{0\}$ for all $x \in K$ and we have (†) in case (iii). However, for autonomous systems a stronger result in case (iii) than (†) is available.

2. Note that in case (iii) not much can be said about the omega limit sets without the additional assumption. When (†) holds, however, then (iii) means that all orbits approach each other and all omega limit sets are identical. This excludes the co-existence of any two constant solutions or any two periodic solutions. On the other hand, eventually all orbits may become periodic or constant, which is depending on the operators T_n; the case of an eventually constant solution will be subject of the next theorem. The most important feature of the limit set trichotomy in Theorem 2 is that the knowledge of one single solution determines whether case (i), (ii) or (iii) holds, and so determines the long-run behavior of all solutions of the system!

3. At this point it is not clear how the additional condition that implies (†) can be verified in applications. Note that $\omega(x, T_n) \subset \mathring{K} \cup \{0\}$ is trivial in the one-dimensional case. However, the crucial condition in Theorem 2 is the uniform contractivity of the sequence of lumped operators $(S_n)_{n \geq 0}$ with respect to the part-metric. When we develop some criteria to check that property in applications in the next section, it turns out that in case (iii) mostly $\lim_{n \to \infty} p(T_n \circ \cdots \circ T_0 x - T_n \circ \cdots \circ T_0 y) = 0$ holds for all $x, y \in K^*$. This, of course, yields (†) directly by Lemma 1, because in (iii) all orbits are bounded in norm.

Where the operators T_n converge to a self-mapping T of K, e.g., for an autonomous discrete dynamical system, an additional statement in case (iii) of Theorem 2 can be given.

THEOREM 3. *Let K be a normal cone in a real Banach space $(E, \|\cdot\|)$ with nonempty interior. Let $(T_n)_{n \geq 0}$ be a sequence of continuous self-mappings of K with $T_n(\mathring{K}) \subset \mathring{K}$ for all $n \geq 0$. Assume further that for an integer $r \geq 1$ the sequence $(S_n)_{n \geq 0}$ of lumped operators $S_n = T_{n+r-1} \circ \cdots \circ T_n$ is uniformly contractive on $\mathring{K}$ for the part-metric and that $S_n(K^*) \subset \mathring{K}$ for all $n \geq 0$. Suppose for the operators T_n uniform convergence on K (with respect to the norm topology) to some continuous operator T with $T(\mathring{K}) \subset \mathring{K}$ and such that $T^s(K^*) \subset \mathring{K}$ for some integer $s \geq 1$. For all bounded orbits $\gamma^+(x, T_n)$, $x \in K$, assume that the closure is compact in K in the norm topology. Then the following trichotomy holds: either*

(i) *for all $x \in K^*$ the orbits $\gamma^+(x, T_n)$ are unbounded in norm; or*

(ii) *for all $x \in K$, $\lim_{n \to \infty} \|T_n \circ \cdots \circ T_0 x\|$; or*

(iii) *there exists some $x^* \in \mathring{K}$ such that $Tx^* = x^*$ and $\lim_{n \to \infty} \|T_n \circ \cdots \circ T_0 x - x^*\| = 0$ for all $x \in K^*$.*

Proof. Obviously, Theorem 2 applies. It follows directly that cases (i) and (ii) of Theorem 3 are implied by the corresponding cases of Theorem 2. To prove case (iii) let $x \in K^*$ be arbitrary but fixed, and let $(x_n)_{n \geq 0}$ be the solution of $(*)$ with $x_0 = x$. Due to Theorem 2 (iii), the orbit $\gamma^+(x, T_n)$ is bounded in norm and, therefore, its closure is compact in K. This implies that $\omega(x, T_n) \neq \emptyset$.

(1) We show that $\omega(x, T_n)$ is compact with respect to the part-metric p and

$$T(\omega(x, T_n)) = \omega(x, T_n) \subset \mathring{K}.$$

(1.1) Let $y \in \omega(x, T_n)$ be arbitrary. Then a subsequence $(x_{k(n)})_{n \geq 0}$ of $(x_n)_{n \geq 0}$ exists that converges to y, and, by taking a further subsequence $(x_{l(n)})_n$, also $(x_{l(n)+1})_{n \geq 0}$ converges to some $z \in \omega(x, T_n)$. Since $T_{l(n)}$ converges to T uniformly on K and T is continuous, it follows that

$$\|Ty - z\| = \lim_{n \to \infty} \left\| Tx_{l(n)} - x_{l(n)+1} \right\| = \lim_{n \to \infty} \left\| Tx_{l(n)} - T_{l(n)} x_{l(n)} \right\| = 0;$$

i.e., $Ty = z$. As $y \in \omega(x, T_n)$ was chosen arbitrary, this implies $T(\omega(x, T_n)) \subset \omega(x, T_n)$. Analogously, we can show that $\omega(x, T_n) \subset T(\omega(x, T_n))$, and we finally have $T(\omega(x, T_n)) = \omega(x, T_n)$. Furthermore, by $T^s(K^*) \subset \mathring{K}$ it follows that $\omega(x, T_n) \subset \mathring{K} \cup \{0\}$. Assume that $0 \in \omega(x, T_n)$, then $T0 = 0$, because $T0 \neq 0$ and $T(\mathring{K}) \subset \mathring{K}$ contradict $T^2(\omega(x, T_n)) = \omega(x, T_n)$. This implies by induction on j that for a subsequence $(x_{k(n)})_{n \geq 0}$ of $(x_n)_{n \geq 0}$ which converges to 0 also the sequences $(x_{k(n)+j})_{n \geq 0}$ converge to 0 for all $j \geq 0$. Hence, it follows that $\omega(x, T_n) = \{0\}$ which is excluded by Theorem 2 (iii). Thus, we conclude that $\omega(x, T_n) \subset \mathring{K}$.

(1.2) Obviously, $\omega(x) = \bigcap_{n \geq 0} \overline{\bigcup_{k \geq n} \{x_k\}}$ is closed in norm as an intersection of closed sets. Hence, by $\omega(x) \subset \overline{\gamma^+(x, T_n)}$, $\omega(x)$ is a closed subset of the compact set $\overline{\gamma^+(x, T_n)}$ and thus compact itself in the norm topology. By applying Lemma 1 one can observe if a set $A \subset \mathring{K}$ is bounded in $(\mathring{K}, p)$, then A is bounded in the norm and the norm closure of A is contained in $\mathring{K}$; and if a set $A \subset \mathring{K}$ is compact in $(\mathring{K}, \|\cdot\|)$, then A is also compact in $(\mathring{K}, p)$. This implies in particular that the norm-compact set $\omega(x, T_n) \subset \mathring{K}$ is also compact in $(\mathring{K}, p)$.

(2) We claim that T^r is contractive on the compact metric space $(Y := \omega(x, T_n), p)$. Then all conditions of [10, Lemma 2.1] are satisfied which implies that $\lim_{n \to \infty} T^n y = x^*$ for all $y \in Y$, where x^* is the unique fixed point of T in Y. Now let $a_0 \in Y$ be arbitrary. Then by $T(Y) = Y$ some $a_1 \in Y$ exists such that $a_0 = Ta_1$. By the same argument we can find some $a_2 \in Y$ such that $a_1 = Ta_2$. Hence, by iteration we obtain for every $n \in \mathbb{N}$ some $a_n \in Y$ such that $a_0 = T^n a_n$. As Y is compact, a subsequence

$(a_{k(n)})_{n\geq 0}$ of $(a_n)_{n\geq 0}$ exists such that $\lim_{n\to\infty} a_{k(n)} = \bar{a} \in Y$. Furthermore, for every $n \in \mathbb{N}$ we find integers $m(n)$ and $j(n) \in \{0, \ldots, r-1\}$ such that $k(n) = m(n) \cdot r + j(n)$. Then the contractivity of T^r yields for all $n \geq 0$,

$$\begin{aligned} p(a_0, x^*) &\leq p\Big(T^{k(n)}a_{k(n)}, T^{k(n)}\bar{a}\Big) + p(T^{k(n)}\bar{a}, x^*) \\ &\leq p\Big(T^{j(n)}a_{k(n)}, T^{j(n)}\bar{a}\Big) + p(T^{k(n)}\bar{a}, x^*) \\ &\leq \max_{i=0,\ldots,r-1} p\Big(T^{i}a_{k(n)}, T^{i}\bar{a}\Big) + p(T^{k(n)}\bar{a}, x^*). \end{aligned}$$

It can be observed by applying Lemma 1 that T is continuous with respect to the part-metric due to the continuity of T in the norm topology and so T^i is continuous for the part-metric for all $i \geq 0$. Since $a_{k(n)}$ converges to $\bar{a}$, and $T^{k(n)}\bar{a}$ converges to x^*, the right hand side of the above-given inequality converges to zero for n tending to infinity. Hence, $p(a_0, x^*) = 0$. This yields $Y = \omega(x, T_n) = \{x^*\}$.

By step (1), also the additional condition in Theorem 2 (iii) is true, and so (†) holds. This implies $\omega(x, T_n) = \omega(y, T_n)$ for all $x, y \in \mathring{K}$. Thus, step (2) yields that $\omega(x, T_n) = \{x^*\}$ for all $x \in \mathring{K}$, where x^* is some fixed point of T in $\mathring{K}$. This concludes the proof. However, what remains is to show that T^r is contractive on $Y := \omega(x, T_n)$ for arbitrary $x \in \mathring{K}$. This will be plain from the following steps. To this end, by applying Lemma 1 again, it can be observed that $T_n \to T$ uniformly on $\mathring{K}$ with respect to the part-metric by the uniform convergence of T_n to T on $\mathring{K}$ in the norm topology.

(3) We show that $(S_n)_{n\geq 0}$ converges uniformly on (Y, p) to $S := T^r$.

(3.1) Suppose that the lumped operators $S'_{k,n} := T_{n+k+1} \circ \cdots \circ T_n$ converge uniformly on (Y, p) to T^k for some integer $k \geq 1$. Then the set

$$U_k = \bigcup_{n=0}^{\infty} S'_{k,n}(Y) \cup Y = \bigcup_{n=0}^{\infty} T_{n+k-1} \circ \cdots \circ T_n(Y) \cup Y$$

is compact in $(\mathring{K}, p)$. To prove this, let $(a_m)_{m\geq 0}$ be an arbitrary sequence in U_k. If $(a_m)_{m\geq 0}$ has any subsequence $(a_{l(m)})_{m\geq 0}$ such that $a_{l(m)} \in S'_{k,n}(Y)$ for all $m \geq 0$ and some integer $n \geq 0$, then $(a_{l(m)})_{m\geq 0}$ has a cluster point in $S'_{k,n}(Y) \subset U_k$, because $S'_{k,n}(Y) = T_{n+k-1} \circ \cdots \circ T_n(Y)$ is compact due to the compactness of Y and the continuity of T_n for all $n \geq 0$. Hence, also $(a_m)_{m\geq 0}$ has a cluster point in U_k. If $(a_m)_{m\geq 0}$ has no such subsequence, increasing sequences $l(m)$ and $n(m)$ exist such that $a_{l(m)} \in S'_{k,n(m)}(Y)$ for all $m \geq 0$. By the uniform convergence of $S'_{k,n}$ to T^k on Y for $n \to \infty$ we find for every $\varepsilon > 0$ some $n_0 \geq 0$ such that

$$p\big(S'_{k,n}y, T^k y\big) < \varepsilon \qquad \text{for all } n \geq n_0, \text{ all } y \in Y.$$

Since $T^k(Y) = Y$ is compact, it follows that for every $y \in Y$ some $z = T^k y \in Y$ exists such that $\lim_{n\to\infty} p(S'_{k,n}y, z) = 0$ and *vice versa*. Thus, $\lim_{n\to\infty} S'_{k,n}(Y) = Y$. Hence, $(a_{l(m)})_{m\geq 0}$ has a cluster point in $Y \subset U_k$. This implies that U_k is compact in $(\mathring{K}, p)$

(3.2) We show by induction that $S'_{k,n}$ converges uniformly on (Y, p) to T^k for $n \to \infty$ for all $k \geq 1$. Since T_n converges uniformly on (Y, p) to the continuous operator T, the set

$$U_1 = \bigcup_{n=0}^{\infty} S'_{1,n}(Y) \cup Y = \bigcup_{n=0}^{\infty} T_n(Y) \cup Y$$

is compact by step (3.1). Thus, T is uniformly continuous on U_1. That is, for every $\varepsilon' > 0$ some $n_0 \geq 0$ exists such that $p(T(Ty), T(T_n y)) \leq \varepsilon'$ for all $n \geq n_0$, all $y \in Y$, because T_n converges to T uniformly on Y and $Ty, T_n y \in U_1$, for all $y \in Y$. Therefore, for every $\varepsilon > 0$ some $n_0 \geq 0$ exists such that

$$p\big(T^2 y, T_{n+1} \circ T_n y\big) \leq p\big(T(Ty), T(T_n y)\big) + p\big(T(T_n y), T_{n+1}(T_n y)\big) \leq \varepsilon$$

for all $n \geq n_0$, all $y \in Y$. Hence $S'_{2,n} = T_{n+1} \circ T_n$ converges uniformly on Y to T^2. Suppose that the lumped operators $S'_{k,n} = T_{n+k-1} \circ \cdots \circ T_n$ converge uniformly on Y to T^k for some integer $k \geq 1$. Then the set $U_k = \bigcup_{n=0}^{\infty} S'_{k,n}(Y) \cup Y$ is compact in X by step (3.1) which yields that T is uniformly continuous on U_k. Taking the uniform convergence of $S'_{k,n}$ to T^k on Y, the uniform convergence of T_{n+k} to T on U_k and the construction of U_k into account, this implies that for every $\varepsilon > 0$ some $n_0 \geq 0$ exists such that

$$\begin{aligned} p\big(T^{k+1}y, S'_{k+1,n}y\big) &\leq p\big(T(T^k y), T(S'_{k,n}y)\big) + p\big(T(S'_{k,n}y), T_{n+k}(S'_{k,n}y)\big) \\ &\leq \varepsilon \end{aligned}$$

for all $n \geq n_0$, all $y \in Y$. Hence $S'_{k+1,n}$ converges uniformly on Y to T^{k+1} which implies by induction that this statement is true for all $k \geq 1$.

(4) Since S_n converges uniformly to $S = T^r$ on (Y, p) by step (3), for every $\varepsilon > 0$ some $n_2 \geq 0$ exists such that $p(S_n y, Sy) \leq \varepsilon$ for all $n \geq n_2$, all $y \in Y$. Hence, by the uniform contractivity of $(S_n)_{n\geq 0}$, for every $\varepsilon > 0$ some $n_0 \geq 0$ exists such that

$$p(Sy, Sz) \leq p(Sy, S_n y) + p(S_n y, S_n z) + p(S_n z, Sz) \leq c(y, z) + 2\varepsilon$$

for all $n \geq n_0$, all $y, z \in Y$. This implies that $p(Sy, Sz) \leq c(y, z) < p(y, z)$ for all $y, z \in Y, y \neq z$. Therefore, $S = T^r$ is contractive on (Y,p). ∎

For the autonomous system $x_{n+1} = Tx_n$ with $T(\mathring{K}) \subset \mathring{K}$ and $T^r(K^*) \subset \mathring{K}$, T^r is contractive on $\mathring{K}$ with respect to the part-metric, Theorem 3 yields the limit set trichotomy for autonomous systems given in [10, Thm. 3.1] as a special case.

3. CONDITIONS FOR CONTRACTIVENESS

What kind of sequences $(T_n)_{n \geq 0}$ of self-mappings T_n of K are contractive with respect to the part-metric? Sufficient conditions for nonexpansiveness of a mapping $T: \mathring{K} \to \mathring{K}$ for a cone K with non-empty interior in a Banach space E are given by the two properties of *monotonicity* and *sublinearity* (subhomogeneity) of T:

$$x, y \in \mathring{K}, \qquad x \leq y \Rightarrow Tx \leq Ty; \tag{a}$$

$$x \in \mathring{K}, \qquad 0 \leq \lambda \leq 1 \Rightarrow \lambda Tx \leq T(\lambda x). \tag{b}$$

It is easy to verify that (a) and (b) imply nonexpansiveness, and that, moreover, T is contractive on $\mathring{K}$ if in addition (b) holds with strict inequality. The following concept from [6] is a strengthening of (a) and (b). The mapping T is called *ascending* on $\mathring{K}$ if there exists a continuous mapping $\varphi : [0, 1] \to [0, 1]$ with $\varphi(\lambda) > \lambda$ for all $0 < \lambda < 1$ and such that

$$x, y \in \mathring{K}, \qquad 0 \leq \lambda \leq 1, \quad \lambda x \leq y \Rightarrow \varphi(\lambda)Tx \leq Ty.$$

Thus, for the case of autonomous systems, that is $T_n = T$ for all $n \geq 0$, an ascending operator T on $\mathring{K}$ is contractive with respect to the part-metric. Furthermore, Krause and Nussbaum [10, p. 865] give detailed conditions for (almost everywhere) differential mappings T on a cone $K \subset \mathbb{R}^n$.

In this section, we shall extend these two conditions to the nonautonomous situation which ensure uniform contractiveness with respect to the part-metric. On the one hand, for sequences $(T_n)_{n \geq 0}$ of operators on normal cones K which are *uniformly ascending* on K; that is, there exists a continuous mapping φ as above such that $\varphi(\lambda)T_n x \leq T_n y$ for all $0 \leq \lambda \leq 1$, all $n \geq 0$ and all $x, y \in K$ with $\lambda x \leq y$. On the other hand, we prove a simple extension of the above mentioned condition in [10] for continuous differentiable mappings T_n on K in the finite dimensional case $E = \mathbb{R}^n$.

LEMMA 4. *Let K be a normal cone in a real Banach space $(E, \|\cdot\|)$ with nonempty interior $\mathring{K}$. Let $(T_n)_{n \geq 0}$ be a sequence of self-mappings of K which is uniformly ascending on K with respect to φ. Then the sequence $(T_n)_{n \geq 0}$ is* *uniformly contractive on* $\overset{\circ}{K}$ *for the part-metric, where the contractivity function* c *is defined for all* $x, y \in \overset{\circ}{K}$ *by*

$$c(x, y) = -\log \varphi(\min\{\lambda(x, y), \lambda(y, x)\}).$$

Proof. Let again $\lambda(x, y) = \sup\{\lambda \in \mathbb{R}_+ : \lambda x \leq y\}$. Then, without loss of generality, we can assume that $\lambda(x, y) \leq 1$ for $x, y \in \overset{\circ}{K}$. Since $\lambda(x, y)x \leq y$ and $(T_n)_{n \geq 0}$ is uniformly ascending, it follows that $\varphi(\lambda(x, y))T_n x \leq T_n y$ for all $n \geq 0$. Therefore, we find that $\lambda(T_n x, T_n y) \geq \varphi(\lambda(x, y))$ for all $n \geq 0$. By $\lambda(x, y) \cdot \lambda(y, x) \leq 1$ for all $x, y \in \overset{\circ}{K}$, it follows that $0 < \min\{\lambda(x, y), \lambda(y, x)\} < 1$ if $x \neq y$. Thus, for all $x, y \in \overset{\circ}{K}$ and all $n \geq 0$ it follows that

$$\varphi(\min\{\lambda(x, y), \lambda(y, x)\}) \leq \min\{\lambda(T_n x, T_n y), \lambda(T_n y, T_n x)\}$$

and moreover, if $x \neq y$,

$$\varphi(\min\{\lambda(x, y), \lambda(y, x)\}) > \min\{\lambda(x, y), \lambda(y, x)\}.$$

Hence, c satisfies both conditions in the definition of uniform contractivity with respect to the part-metric. Now let $x, y \in \overset{\circ}{K}$ be arbitrary. Then for every $\varepsilon > 0$ and $x' \in \overset{\circ}{K}$ such that $p(x, x') < \varepsilon$ it follows that

$$e^{-c(x, x')} = \varphi(e^{-p(x, x')}) > e^{-\varepsilon}$$

and thus $c(x, x') < \varepsilon$. Therefore, $|c(x, y) - c(x', y)| \leq c(x, x') < \varepsilon$ and so, by the symmetry of c, we obtain that c is continuous in both components with respect to the part-metric. Hence, $(T_n)_{n \geq 0}$ is uniformly contractive on $\overset{\circ}{K}$ with respect to the part-metric. ▮

With the result of Lemma 4 we can state the following corollary of Theorem 2.

COROLLARY 5. *Let* K *be a normal cone in a real Banach space* $(E, \|\cdot\|)$ *with nonempty interior* $\overset{\circ}{K}$, *and let* $(T_n)_{n \geq 0}$ *be a sequence of continuous self-mappings of* K *with* $T_n(\overset{\circ}{K}) \subset \overset{\circ}{K}$ *for all* $n \geq 0$. *For all bounded orbits* $\gamma^+(x, T_n)$, $x \in K$, *assume that the closure is compact in* K. *If for some integer* $r \geq 1$ *the sequence* $(S_n)_{n \geq 0}$ *of lumped operators is uniformly ascending on* K *with* $S_n(K^*) \subset \overset{\circ}{K}$ *for all* $n \geq 0$, *then the following limit set trichotomy holds for the nonautonomous system* $(*)$: *either*

$$\left.\begin{array}{ll} \text{(i)} & \textit{for all } x \in K^* \textit{ the orbits } \gamma^+(x, T_n) \textit{ are unbounded in norm; or} \\ \text{(ii)} & \textit{for all } x \in K, \lim_{n\to\infty} \|T_n \circ \cdots \circ T_0 x\| = 0; \textit{ or} \\ \text{(iii)} & \textit{for all } x \in K^* \textit{ the orbits } \gamma^+(x, T_n) \textit{ are bounded in norm, have a} \\ & \textit{non-trivial cluster point, and} \\ & \displaystyle\lim_{n\to\infty} \|T_n \circ \cdots \circ T_0 x - T_n \circ \cdots \circ T_0 y\| = 0 \qquad \textit{for all } x, y \in K^*. \end{array}\right\} \quad (\S)$$

Proof. By Lemma 4 all conditions of Theorem 2 are satisfied. To obtain (§) we show that in case (iii),

$$\lim_{n\to\infty} p(T_n \circ \cdots \circ T_0 x, T_n \circ \cdots \circ T_0 y) = 0 \qquad \text{for all } x, y \in K^*. \qquad (¶)$$

This implies (iii) in (§) by Lemma 1, because all orbits are bounded in norm.

To prove (¶), let $(x_n)_{n\geq 0}$ and $(y_n)_{n\geq 0}$ be two arbitrary solutions of $(*)$ with initial conditions $x_0, y_0 \in K^*$, and assume that neither (i) nor (ii) holds. Then, the orbits $\gamma^+(x_0, T_n)$ and $\gamma^+(y_0, T_n)$ are bounded in norm. Furthermore, by $S_n(K^*) \subset \mathring{K}$ and $T_n(\mathring{K}) \subset \mathring{K}$ for all $n \geq 0$ we have $x_n, y_n \in \mathring{K}$ for all $n \geq r$. This implies that $\Gamma := \{(x_n, y_n): n \geq r\} \subset \mathring{K} \times \mathring{K}$ is relatively compact. Since $(S_n)_{n\geq 0}$ is uniformly contractive on $\mathring{K}$ by Lemma 4, it follows that

$$p(x_n, y_n) \geq c(x_n, y_n) \geq p(x_{n+r}, y_{n+r}) \geq p^*$$

for all $n \geq r$ and some $p^* \geq 0$. In particular, if (¶) is not true, it follows that $\hat{p} \geq p(x, y) \geq p^* > 0$ for all $(x, y) \in \Gamma$ and some $\hat{p} < \infty$. Letting $\lambda_{\min}(x, y) = e^{-p(x,y)}$ for all $(x, y) \in \Gamma$, we obtain $0 < e^{-\hat{p}} \leq \lambda_{\min}(x, y) \leq e^{-p^*} < 1$ for all $(x, y) \in \Gamma$ and so the continuity of φ implies that

$$\frac{\varphi(\lambda_{\min}(x, y))}{\lambda_{\min}(x, y)} \geq \alpha' > 1 \qquad \text{for all } (x, y) \in \Gamma.$$

Hence, $p^* > 0$ implies the contradiction

$$\begin{aligned} p^* &\leq p(x_{(n+1)r}, y_{(n+1)r}) \leq c(x_{nr}, y_{nr}) = -\log \varphi(\lambda_{\min}(x_{nr}, y_{nr})) \\ &\leq -\log(\alpha' \cdot \lambda_{\min}(x_{nr}, y_{nr})) = p(x_{nr}, y_{nr}) - \alpha \\ &\vdots \\ &\leq p(x_r, y_r) - n \cdot \alpha \end{aligned}$$

for all $n \geq 1$, where $\alpha = \log \alpha' > 0$. Thus, we must have $p^* = 0$ and so (¶) holds. ∎

Remark. In [13] it is shown that under the conditions of Corollary 5 additional statements can be given for case (i).

EXAMPLES. 1. Let $E = C([0,1])$ be the real continuous functions on $[0,1]$ equipped with the sup-norm, and let $K = C_+([0,1])$ be the non-negative functions in E. For $x(\cdot) \in K$, all $n \geq 0$ define the nonlinear integral operators

$$(T_n x)(\sigma) = \int_0^1 k_n(\sigma, \tau) f(x(\tau))\, d\tau,$$

where $f \in K$ is an (uniformly) ascending function such that $f(x) > 0$ for $x > 0$. The kernels k_n are assumed to be continuous, non-negative and not identically zero on $[0,1]^2$. Examples that imply uniformly ascending operators T_n are $f(x) = x^p$, $0 < p < 1$, with $\varphi(\lambda) = \lambda^p$ and $f(x) = (ax + c)/(bx + d)$, $0 < bc \leq ad$, with $\varphi(\lambda) = \lambda + (bc)/(ad)(1 - \lambda)$. By Lemma 4, in both examples the sequence $(T_n)_{n \geq 0}$ is uniformly contractive with respect to the part-metric. Corollary 5 applies to these examples for $r = 1$ which states limit set trichotomy for this system.

2. Let $E = \mathbb{R}^2$ and $K = \mathbb{R}^2_+$ the positive orthant of $\mathbb{R}^2$. Consider the difference equation of second order

$$u_{n+2} = a_n\sqrt{u_{n+1}} + b_n\sqrt{u_n} + c_n \qquad \text{for } n \geq 0,$$

where $a_n, b_n, c_n \geq 0$ such that $a_n \cdot b_n + c_n > 0$ for all $n \geq 0$. Letting $x_n = [u_n, u_{n+1}]^T$, we have the equivalent discrete dynamical system $x_{n+1} = T_n x_n$, where the nonlinear operator T_n is defined by

$$T_n x = \begin{bmatrix} x_2 \\ a_n\sqrt{x_2} + b_n\sqrt{x_1} + c_n \end{bmatrix}.$$

Corollary 5 applies to this example for $r = 2$, because the sequence of lumped operators $(S_n)_{n \geq 0}$ is uniformly ascending for $\varphi(\lambda) = \lambda^{1/2}$. Hence, for this difference equation limit set trichotomy holds. For the special case $a_n \to a$, $b_n \to b$ and $c_n \to 0$ for $n \to \infty$, Theorem 3 yields that $u^* = (a + b)^2$ is globally asymptotical stable for this difference equation. By $T_{n+1} \circ T_n x \geq [c_n, c_{n+1}]^T$ for all $n \geq 0$, all $x \in K$ it follows that case (i) of the limit set trichotomy holds if $(c_n)_{n \geq 0}$ is an unbounded sequence. Finally, it can be observed that case (ii) holds if all of the three parameters converge to zero for $n \to \infty$.

In the following, we consider the special case $E = \mathbb{R}^k$ with a cone $K \subset \mathbb{R}^k_+$. For $x, y \in \mathbb{R}^k_+$ and $0 \leq \tau \leq 1$ we write $x^{1-\tau}y^\tau$ for the vector with components $x_i^{1-\tau}y_i^\tau$, $i = 1, \ldots, k$. Similarly, for $x \in \mathring{K}$ define $\exp(x)$ and $\log(x)$ coordinate-wise. In imitation of [10] we call a subset G of $\mathring{K}$ *logarithmically convex* if $x, y \in G$ implies $x^{1-\tau}y^\tau \in G$ for all $0 < \tau < 1$. Furthermore, let T_i denote the ith coordinate function of T for $i = 1, \ldots, k$.

LEMMA 6. *Let G be an open, logarithmically convex subset of a cone $K \subset \mathbb{R}^k_+$. Let $T\colon G \to \mathring{K}$ be continuously differentiable and let $\alpha\colon G \to \mathbb{R}_+$*

be continuous such that

$$\sum_{j=1}^{k} x_j \left| \frac{\partial T_i}{\partial x_j}(x) \right| \le \alpha(x) \cdot T_i x \qquad \textit{for } i = 1, \dots, k \textit{ and all } x \in G. \quad (**)$$

Then it is true that

$$p(Tx, Ty) \le \sup_{0 \le \tau \le 1} \alpha\left(x^{\tau} y^{1-\tau}\right) \cdot p(x, y)$$

for all $x, y \in G$.

Proof. (1) Let $f\colon G' \to K$ be a continuously differentiable mapping in $(\mathbb{R}^k, \|\cdot\|_\infty)$, where $G' = \log(G)$ and $\|\cdot\|_\infty$ denotes the sup-norm. Obviously, $G' \subset \mathbb{R}^k$ is a convex set. Then, by the mean value Theorem, for $x, y \in G'$ we find that

$$f(y) - f(x) = \left(\int_0^1 Df((1-\tau)x + \tau y)\, d\tau \right) \cdot (y - x),$$

where $Df(z)$ denotes the Jacobian-matrix of f in $z \in G'$ and the integral is taken coordinate-wise. If a matrix $A = (a_{ij})_{ij} \in \mathbb{R}^{k \times k}$ is identified with a linear self-mapping of $\mathbb{R}^k$, the sup-norm of A is given by $\|A\|_\infty = \max_{i=1,\dots,k} \sum_{j=1}^{k} |a_{ij}|$. Hence, for $\alpha'(x) := \max_{i=1,\dots,k} \sum_{j=1}^{k} |\partial f_i(x)/\partial x_j|$ we obtain

$$\begin{aligned}
\| f(y) - f(x) \|_\infty &\le \left(\int_0^1 \| Df((1-\tau)x + \tau y) \|_\infty \, d\tau \right) \cdot \|y - x\|_\infty \\
&= \left(\int_0^1 \max_{i=1,\dots,k} \sum_{j=1}^{k} \left| \frac{\partial f_i((1-\tau)x + \tau y)}{\partial\big((1-\tau)x_j + \tau y_j\big)} \right| d\tau \right) \cdot \|y - x\|_\infty \\
&= \left(\int_0^1 \alpha'((1-\tau)x + \tau y)\, d\tau \right) \cdot \|y - x\|_\infty \\
&\le \sup_{0 \le \tau \le 1} \alpha'((1-\tau)x + \tau y) \cdot \|y - x\|_\infty
\end{aligned}$$

for all $x, y \in G'$.

(2) One verifies easily that on $\mathring{K} \subset \mathbb{R}^k_+$ the part-metric becomes

$$p(x, y) = \max\left\{ |\log y_j - \log x_j| \colon j = 1, \dots, k \right\} \qquad \text{for all } x, y \in \mathring{K},$$

and that the complete metric spaces $(\mathring{K}, p)$ and $(\mathbb{R}^k, \|\cdot\|_\infty)$ are isometric for the isometry $\phi(x) = \log(x)$, $\phi^{-1}(y) = \exp(y)$. It follows that the assertion is proved when we show that

$$\|\log(Ty) - \log(Tx)\|_\infty \le \sup_{0\le\tau\le 1} \alpha\left(x^\tau y^{1-\tau}\right) \cdot \|\log y - \log x\|_\infty$$

for all $x, y \in G$. If we define f by $f(x) = \log(T(\exp x))$ for $x \in G$, then it follows that $f\colon G' \to K$ and

$$\frac{\partial f_i(x)}{\partial x_j} = \frac{\exp x_j}{T_i(\exp x)} \cdot \frac{\partial T_i(\exp x)}{\partial(\exp x_j)}.$$

Hence, from step (1) follows that

$$\begin{aligned}\|f(\log y) - f(\log x)\|_\infty \le \sup_{0\le\tau\le 1} &\alpha'((1-\tau)\log x + \tau\log y)\\ &\cdot\|\log y - \log x\|_\infty\end{aligned}$$

for all $x, y \in G$, where

$$\alpha'(\log x) = \max_{i=1,\dots,k} \sum_{j=1}^{k} \left|\frac{\partial f_i(\log x)}{\partial(\log x_j)}\right| = \max_{i=1,\dots,k} \sum_{j=1}^{k} \frac{x_j}{T_i x} \cdot \left|\frac{\partial T_i x}{\partial x_j}\right| \le \alpha(x).$$

Thus, $\alpha'((1-\tau)\log x + \tau\log y) \le \alpha(x^\tau y^{1-\tau})$, which yields the assertion. ∎

From Lemma 6 it follows that the operator T in Lemma 6 is contractive with respect to the part-metric on G if $\alpha(x) < 1$ for all $x \in G$. If, in addition, $\alpha(x) \le \rho < 1$ for all $x \in G$, then T is a contraction on G with respect to the part-metric. We conclude the following corollary of Theorem 2.

COROLLARY 7. *Let K be a normal cone in $(\mathbb{R}^k_+, \|\cdot\|)$ with nonempty interior $\mathring{K}$, and let $(T_n)_{n\ge 0}$ be a sequence of continuously differentiable self-mappings of K with $T_n(\mathring{K}) \subset \mathring{K}$ for all $n \ge 0$. Suppose that some integer $r \ge 1$ for the sequence $(S_n)_{n\ge 0}$ of lumped operators it is true that $S_n(K^*) \subset \mathring{K}$ and*

$$\sum_{j=1}^{k} x_j \left|\frac{\partial S_{n,i}x}{\partial x_j}\right| \le \alpha(x)\cdot S_{n,i}x \qquad (S_{n,i}x \text{ the } i\text{th coordinate}) \qquad (\dagger\dagger)$$

for $i = 1, \dots, k$ and all $n \ge 0$, all $x \in \mathring{K}$, where $\alpha\colon K^* \to \mathbb{R}_+$ is a continuous mapping such that $\alpha(x) < 1$ for all $x \in K^*$. Then limit set trichotomy (§) holds for the nonautonomous system $(*)$.

Proof. Define for all $x, y \in K^*$ with $x \sim y$ the mapping

$$c(x, y) = \sup_{0 \leq \tau \leq 1} \alpha\left(x^{\tau} y^{1-\tau}\right) \cdot p(x, y).$$

By assumption we have $c(x, y) < p(x, y)$ for all $x, y \in K^*$ with $x \neq y, x \sim y$. Since $\mathring{K}$ is an open, logarithmically convex set, we also have $p(S_n x, S_n y) \leq c(x, y)$ for all $x, y \in \mathring{K}$, all $n \geq 0$ by Lemma 6. Clearly, c is continuous with respect to the part-metric and so the sequence $(S_n)_{n \geq 0}$ is uniformly contractive on $\mathring{K}$ with respect to the part-metric. Thus, Theorem 2 applies and so it remains to show the additional statement in case (iii) of (§).

To this end, let $(x_n)_{n \geq 0}$ and $(y_n)_{n \geq 0}$ be two arbitrary solutions of $(*)$ with initial conditions $x_0, y_0 \in K^*$, and assume that neither (i) nor (ii) holds. Then the orbits $\gamma^+(x_0, T_n)$ and $\gamma^+(y_0, T_n)$ are bounded in norm. Furthermore, by $S_n(K^*) \subset \mathring{K}$ and $T_n(\mathring{K}) \subset \mathring{K}$ for all $n \geq 0$ we have $x_n, y_n \in \mathring{K}$ for all $n \geq r$. This implies that

$$p(x_{n+r}, y_{n+r}) = p(S_n x_n, S_n y_n) \leq c(x_n, y_n) \leq p(x_n, y_n) \qquad \text{for all } n \geq r,$$

and thus $p(x_n, y_n) \leq \max\{p(x_i, y_i) : i = r, \ldots, 2r - 1\} < \infty$ for all $n \geq r$. Hence, $x_n \sim y_n$ for all $n \geq r$. Let now $J \subset \mathbb{N}$ denote an infinite set of indices such that the subsequence $(x_n)_{n \geq 0,\, n \in J}$ of $(x_n)_{n \geq 0}$ only has the trivial cluster point 0; and let $J = \varnothing$ if $(x_n)_{n \geq 0}$ has no trivial cluster point. By $p(x_n, y_n) \leq \max\{p(x_i, y_i) : i = r, \ldots, 2r - 1\} < \infty$ for all $n \geq r$ we can find some $\lambda > 0$ such that $\lambda x_n \leq y_n \leq \lambda^{-1} x_n$ for all $n \in J$, $n \geq r$. Together with the boundedness of the two solutions, this implies

$$\lim_{\substack{n \to \infty \\ n \in J}} (x_n, y_n) = (0, 0).$$

So it remains to show that $\lim_{n \to \infty, n \notin J} \|x_n - y_n\| = 0$. Note that by the construction of J the closure of $\Gamma := \{(x_n, y_n) \colon n \geq r, n \notin J\} \subset \mathring{K} \times \mathring{K}$ is compact in $K^* \times K^*$ in norm; thus

$$\sup_{0 \leq \tau \leq 1} \alpha\left(x^{\tau} y^{1-\tau}\right) \leq \alpha < 1 \qquad \text{for all } (x, y) \in \Gamma.$$

This implies that

$$p(x_{n+r}, y_{n+r}) = p(S_n x_n, S_n y_n) \leq c(x_n, y_n) \leq \alpha \cdot p(x_n, y_n)$$

for all $n \geq r$, $n \notin J$. Since for every $n \in \mathbb{N}$ we can find $m \in \mathbb{N}$ and $0 \leq j \leq r - 1$ such that $n = mr + j$, we obtain by $p(x_r, y_r) < \infty$ and by $\alpha < 1$,

$$0 \leq \lim_{\substack{m \to \infty \\ (mr+j) \notin J}} p(x_{mr+j}, y_{mr+j}) \leq \lim_{\substack{m \to \infty \\ (mr+j) \notin J}} \alpha^{m-1} p(x_{r+j}, y_{r+j}) = 0$$

for all $j = 0, \ldots, r-1$. Taking the compactness of $\overline{\Gamma}$ in the norm topology into account, Lemma 1 implies

$$\lim_{\substack{n \to \infty \\ n \notin J}} \|x_n - y_n\| = 0.$$

This concludes the proof. ∎

EXAMPLE. Let $E = \mathbb{R}$, $K = \mathbb{R}_+$ and $T_n x = a_n x^{1/2}$ with $a_n \geq 0$ for all $n \geq 0$. For $r = 1$ we have $S_n = T_n$ and

$$x \cdot |T_n' x| = \tfrac{1}{2} \cdot T_n x$$

for all $n \geq 0$. So we can choose $\alpha(x) = \frac{1}{2}$ and thus $(T_n)_{n \geq 0}$ is contractive on $\mathring{K}$ with respect to the part-metric. Hence, Corollary 7 applies to this system and yields limit set trichotomy.

REFERENCES

1. H. Bauer and H. S. Bear, The part metric in convex sets, *Pacific J. Math.* **30** (1969), 15–33.
2. J. Cohen, Ergodic theorems in demography, *Bull. Amer. Math. Soc.* **1** (1979), 275–295.
3. T. Fujimoto and U. Krause, Asymptotic properties for inhomogeneous iterations of nonlinear operators, *SIAM J. Math. Anal.* **19** (1988), No. 4, 841–853.
4. V. L. Kocic and G. Ladas, "Global Behavior of Nonlinear Difference Equations of Higher-Order with Applications," Kluwer Academic, Dordrecht, 1993.
5. V. L. Kocic, G. Ladas, and I. W. Rodrigues, On rational recursive sequences, *J. Math. Anal. Appl.* **173** (1993), 127–157.
6. U. Krause, A nonlinear extension of the Birkhoff–Jentzsch theorem, *J. Math. Anal. Appl.* **114** (1986), 552–568.
7. U. Krause, Perron's stability theorem for non-linear mappings, *J. Math. Econom.* **15** (1986), 275–282.
8. U. Krause, Stability trichotomy, path stability, and relative stability for positive non-linear difference equations of higher order, *J. Differ. Equations Appl.* **1** (1995), 323–346.
9. U. Krause, Positive nonlinear systems: Some results and applications, Proceedings of the First World Congress of Nonlinear Analysts, New York (V. Lakshmikantham, Ed.), 1996, pp. 1529–1539.
10. U. Krause and R. D. Nussbaum, A limit set trichotomy for self-mappings of normal cones in Banach spaces, *Nonlinear Anal. TMA* **20** (1993), No. 7, 855–870.
11. U. Krause and P. Ranft, A limit set trichotomy for monotone nonlinear dynamical systems, *Nonlinear Anal. TMA* **19** (1992), 375–392.
12. T. Nesemann, An application of a nonlinear extension of the Coale-Lopez theorem, IMACS World Congress (15th) on Scientific Computation, Modelling, and Applied Mathematics, (A. Sydow, Ed.), Vol. 1, pp. 19–24, Wissenschaft u. Techn., Berlin, 1997.
13. T. Nesemann, A nonlinear extension of the Coale-Lopez theorem, *Positivity* (1998), to appear.
14. R. D. Nussbaum, Hilbert's projective metric and iterated nonlinear maps, *Mem. Amer. Math. Soc.* **75** (1988), no. 391.

15. J. H. Pollard, ‘‘Mathematical Models for the Growth of Human Populations,’’ Cambridge Univ. Press, Cambridge, UK, 1973.
16. H. H. Schaefer, ‘‘Topological Vector Spaces,’’ Springer-Verlag, Berlin, 1971.
17. E. Seneta, ‘‘Non-negative Matrices and Markov Chains,’’ 2nd ed., Springer-Verlag, Berlin, 1980.
18. H. L. Smith, Cooperative systems of differential equations with concave nonlinearities, *Nonlinear Anal. TMA* **10** (1986), 1037–1052.
19. P. Takàč, Asymptotic behavior of discrete time semigroups of sublinear, strongly increasing mappings with applications to biology, *Nonlinear Anal. TMA* **14** (1990), 35–42.
20. A. C. Thompson, On certain contraction mappings on a partially ordered vector space, *Proc. Amer. Math. Soc.* **14** (1963), 438–443.